On the Role of Chapman's Hydrostatic Solar Wind Mechanism in Parker's Hydrodynamic Solar Wind Model

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Abstract

The *global* role of Chapman's hydrostatic solar wind mechanism [4] in Parker's hydrodynamic solar wind model [6] is investigated by using the *de Laval nozzle* analogy (Clauser [14], Parker [15]) for the latter model. The action of solar gravity in Parker's hydrodynamic solar wind model is shown to be geometrically equivalent to a *renormalization* of the wind channel area, which is described precisely by Chapman's hydrostatic density profile [4]. So, Chapman's hydrostatic solar wind mechanism [4] appears to continue to be operative, on a *global* level (not just locally near the coronal base), in Parker's hydrodynamic solar wind model [6], the effects of solar gravity in Parker's hydrodynamic model [6] being *essentially* encapsulated by Chapman's hydrostatic model [4]. This result is shown to be robust by considering both isothermal gas and polytropic gas models for the solar wind.

1 Introduction

The solar wind is a hot tenuous plasma outflowing continually from the sun, which carries off a huge amount of angular momentum from the sun while inflicting only a negligible loss of mass (Meyer-Vernet [1]). The bulk of the solar wind is known to emerge from the coronal holes (Sakao et al. [2]), and to fill the heliosphere (Dialynas et al. [3]). Weak to moderate speed solar wind is believed to be caused by coronal heating along with high thermal conduction. Chapman [4] argued that the corona is governed by a near hydrostatic force balance condition due to the strong binding of the corona by solar gravity, and hence gave a *hydrostatic* model for the static corona. Lamers and Casinelli [5] numerically demonstrated that the corona is almost in hydrostatic equilibrium not just at its base, but until close to the *Parker sonic critical point* ($r = r_*$), where the wind flow speed equals the sound speed in the wind.

However, away from the sun, as Parker [6] pioneeringly pointed out, the thermal energy of the corona greatly exceeds the gravitational binding energy, so Chapman's [4] static corona model would become inaccurate, and the radial coronal flow is no longer negligible. Parker [6] gave an ingenious stationary *hydrodynamic* model for the solar wind, which enables the solar wind to accelerate *continuously* from subsonic speeds at the coronal base to supersonic speeds away from the sun *via* conversion of the thermal energy in the wind beyond the coronal base into kinetic energy of the outward flow. Solar wind observations (Schrijver [7]) indicated that the large-scale behavior of the solar wind, on the average, its local noisiness (Feldman et al. [8]) notwithstanding, is apparently close to Parker's steady solar wind solution[1].

On the other hand, the continuous acceleration of the solar wind to supersonic speeds, as described by Parker's hydrodynamic solar wind model [6], led to the surmise of a *de Laval nozzle* type mechanism (Clauser [14], Parker [15]) implicit in the latter model. The effective de Laval nozzle associated with Parker's hydrodynamic solar wind model was also shown (Shivamoggi [13]) to have a minimum cross-section area at the Parker sonic critical point ($r = r_*$), as expected.

Near the coronal base ($r \ll r_*$), Parker's steady solar wind solution [6] reduces as, expected, to Chapman's hydrostatic solar wind solution [4]. However, the numerical calculations of Lamers and Casinelli [5] showed that the density profiles given by Chapman's hydrostatic model [4] are almost identical to those given by Parker's hydrodynamic model [6] (corresponding to the same temperature) in the *whole* subcritical region ($r < r_*$). The purpose of this paper is therefore to investigate the *global* role of Chapman's hydrostatic solar wind mechanism [4] in Parker's hydrodynamic solar wind model [6]. This is accomplished by using the de Laval nozzle analogy (Clauser [14], Parker [15]) for Parker's hydrodynamic solar wind model [6]. The robustness of this development is confirmed by considering both isothermal gas and polytropic gas models (Parker [16], Holzer [17], Shivamoggi and Pohl [18]) for the solar wind.

2 Parker's Hydrodynamic Solar Wind Model

Consider an ideal gas radial flow constituting the solar wind emanating from a central gravitating point mass representing the sun (Parker [6]). The flow is assumed to be steady and spherically symmetric so that the flow variables depend only on the distance r from the sun. Consider the gas flow to occur in a stream tube of cross-sectional area $\mathrm{A}(r) = 4\pi r^2$ under isothermal conditions.

[1]The Parker Solar Probe (Shivamoggi [9]) has been providing significant information on the conditions in the inner solar corona (Fisk and Casper [10], Bowen et al. [11], and others) some of which were at variance with previous

The equations expressing the conservation of mass and momentum balance for this gas flow are (in usual notations),

$$\frac{1}{\rho}\frac{d\rho}{dr} + \frac{1}{V_r}\frac{dV_r}{dr} + \frac{1}{A}\frac{dA}{dr} = 0 \tag{1}$$

$$\rho V_r \frac{dV_r}{dr} = -\frac{dp}{dr} - \rho\frac{GM_s}{r^2} \tag{2}$$

where G is the gravitational constant and M_s is the mass of the sun, and p and ρ are related to each other *via* the isothermal gas equation of state,

$$p = a^2\rho \tag{3}$$

a being the constant speed of sound in the gas. We assume that the flow variables, as well as their derivatives, vary continuously, so there are no shocks occurring anywhere in the region under consideration.

Equations (1) – (3) lead to:

$$\frac{1}{V_r}\left(\frac{V_r^2}{a^2} - 1\right)\frac{dV_r}{dr} = \left(\frac{A'}{A} - \frac{2r_*}{r^2}\right) \tag{4}$$

where $r_* \equiv \frac{GM_s}{2a^2}$ locates the *Parker sonic critical point,* where the wind flow speed equals the speed of sound in the wind.

On noting $\mathrm{A}(r) = 4\pi r^2$, equation (4) becomes,

$$\frac{1}{V_r}\left(\frac{V_r^2}{a^2} - 1\right)\frac{dV_r}{dr} = \frac{2}{r^2}(r - r_*) \tag{5}$$

which indicates the acceleration of subsonic wind flows to sonic speeds for $r < r_*$, and the acceleration of wind flows to supersonic speeds for $r > r_*$.

3 The de Laval Nozzle Analogy

The *continuous* acceleration of the solar wind, as described by Parker's hydrodynamic solar wind model [10], from subsonic speeds at the coronal base to supersonic speeds away from the sun led to the surmise of a *de Laval nozzle* type mechanism (Clauser [14], Parker [15]) implicit in Parker's hydrodynamic solar wind model [6].

Indeed, if $\mathcal{A} = \mathcal{A}(r)$ is the cross-sectional area of the *effective de Laval nozzle* associated with Parker's solar wind model, equation (4) leads to

$$\frac{1}{V_r}\left(\frac{V_r^2}{a^2} - 1\right)\frac{dV_r}{dr} = \left(\frac{A'}{A} - \frac{2r_*}{r^2}\right) \equiv \frac{\mathcal{A}'}{\mathcal{A}}. \tag{6}$$

belief (like the coupling of the solar wind with solar rotation (Kasper et al. [12]) which was shown (Shivamoggi [13]) to cause enhanced angular momentum loss from the sun).

On using the boundary condition at the surface of the sun, given by $r = r_0$,

$$r = r_0 : \mathcal{A} = A \tag{7}$$

equation (6) gives,

$$\mathcal{A}(r) = A(r) e^{\frac{2r_*}{r_0}\left(\frac{r_0}{r} - 1\right)}. \tag{8}$$

(8) indicates a recipe to *renormalize* the actual wind channel cross-sectional $A(r)$, incorporating the solar gravity geometrically *via* a multiplicative correction to yield the effective de Laval nozzle cross-sectional area $\mathcal{A}(r)$. On the other hand, equation (6) may be viewed alternatively as an *ansatz* to effectively excise solar gravity out of Parker's hydrodynamic solar wind model [10].

It may be noted that far away from the sun, (8) yields,

$$\mathcal{A}(r) \approx A(r) e^{-\frac{2r_*}{r_0}} \tag{9}$$

which indicates that the effective de Laval nozzle cross-sectional area $\mathcal{A}(r)$ increases like the actual wind-channel area $A(r)$ far away from the sun, where the solar gravity becomes unimportant, as to be expected.

Furthermore, putting $A(r) = 4\pi r^2$, we have from (8),

$$\mathcal{A}'(r) = 8\pi (r - r_*) e^{\frac{2r_*}{r_0}\left(\frac{r_0}{r} - 1\right)} \tag{10}$$

which yields,

$$\mathcal{A}'(r) \lesseqgtr 0, \quad \text{if} \quad r \lesseqgtr r_*. \tag{11}$$

In addition, noting from (10),

$$\mathcal{A}''(r) = 8\pi \left[1 - \frac{2r_*}{r^2}(r - r_*)\right] e^{\frac{2r_*}{r_0}\left(\frac{r_0}{r} - 1\right)} \tag{12}$$

we have,

$$r \approx r_* : \quad \mathcal{A}(r) = \mathcal{A}(r_*) + \frac{1}{2}\mathcal{A}''(r_*)(\Delta r)^2 = e^{2(1 - r_*/r_0)}\left[r_*^2 + 4\pi(\Delta r)^2\right] \tag{13}$$

where,

$$\Delta r \equiv r - r_*.$$

(11) and (13) both confirm that the effective de Laval nozzle exhibits a minimum cross-sectional area at the Parker's sonic critical point, as to be expected.

Interesting physical implications of these results ensue by noting that the *hydrostatic force balance* from equation (2), on using equation (3), gives (Chapman [4]),

$$-a^2 \frac{d\rho_h}{dr} - \frac{GM_s}{r^2}\rho_h = 0 \tag{14}$$

On using the boundary condition at the surface of the sun,

$$r = r_0 : \quad \rho = \rho_0 \tag{15}$$

equation (14) yields

$$\rho_h = \rho_0 e^{\frac{2r_*}{r_0}\left(\frac{r_0}{r}-1\right)} \tag{16}$$

Using (16), (8) may be rewritten as,

$$\mathcal{A}(r) = A(r)\left[\frac{\rho_h(r)}{\rho_0}\right]. \tag{17}$$

(17) shows that the correction factor needed to *renormalize* the actual wind-channel area, to incorporate the solar gravity geometrically, is precisely the *Chapman hydrostatic density profile* ρ_h/ρ_0. So, Chapman's hydrostatic solar wind mechanism [4] appears to continue to be operative, on a *global* level, Parker's hydrodynamic solar wind model [6], and as (17) indicates, the effects of solar gravity in Parker's hydrodynamic model [6] are essentially encapsulated by Chapman's hydrostatic model [4]. Numerical calculations (Lamers and Cassinelli [5]), indeed demonstrated that the corona is almost in hydrostatic equilibrium not just at its base, but until close to the Parker sonic critical point, so the density profiles associated with Chapman's hydrostatic model [4] and Parker's hydrodynamic model [6] are almost identical in the *whole* subcritical region ($r \leq r_*$).

4 Polytropic Gas Effects on the de Laval Nozzle Analogy

The solar wind has been found (Boldyrev et al. [19]) not to cool down as fast as that caused by an adiabatic expansion. This may be traced to significant heating occurring in the corona, impairing adiabaticity in the wind. This situation can be dealt with by using a polytropic gas model, described by

$$p = C\rho^\gamma \tag{18}$$

where γ is the polytropic exponent, $1 < \gamma < 5/3$, and C is an arbitrary constant (Parker [16], Holzer [17], Shivamoggi and Pohl [18]). The polytropic exponent γ characterizes the extent to which the solar coronal gas conditions deviate from adiabatic conditions ($\gamma = 5/3$).

The variations in the sound speed a, given by,

$$a^2 = \frac{dp}{d\rho} \tag{19}$$

for a polytropic gas, on using the conservation of the total energy in the solar wind gas,

$$E \equiv \frac{v_r^2}{2} + \frac{a^2}{\gamma-1} - \frac{GM_s}{r} = \text{const} \equiv \frac{a_0^2}{\gamma-1}, \tag{20}$$

are described by (Shivamoggi and Pohl [18]),

$$\frac{a^2}{a_0^2} = \frac{1+4\alpha\frac{r_{*0}}{r}}{1+\alpha M^2} \tag{21}$$

where M is the Mach number of the flow,

$$M \equiv \frac{V_r}{a} \tag{22}$$

and

$$r_{*0} \equiv \frac{GM_s}{2a_0^2}, \quad \alpha \equiv \frac{\gamma - 1}{2}. \tag{23}$$

On writing equation (4) as,

$$\frac{1}{V_r}\left(\frac{V_r^2}{a^2} - 1\right)\frac{dV_r}{dr} = \left[\frac{A'}{A} - \frac{2}{r^2}\left(\frac{r_*}{r_{*0}}\right) r_{*0}\right], \tag{24}$$

and noting,

$$\frac{r_*}{r_{*0}} = \frac{a_0^2}{a^2} \tag{25}$$

and using (21), equation (24) becomes

$$\frac{1}{V_r}\left(\frac{V_r^2}{a^2} - 1\right)\frac{dV_r}{dr} = \left[\frac{A'}{A} - \frac{2r_{*0}}{r^2}\left\{\frac{1 + \alpha M^2}{1 + 4\alpha\frac{r_{*0}}{r}}\right\}\right] \tag{26}$$

If $\mathcal{A} = \mathcal{A}(r)$ is the cross-sectional area of the effective de Laval nozzle associated with Parker's hydrodynamic polytropic solar wind model, we have

$$\frac{1}{V_r}\left(\frac{V_r^2}{a^2} - 1\right)\frac{dV_r}{dr} = \frac{\mathcal{A}'}{\mathcal{A}}.$$

equation (26) then leads to

$$\frac{\mathcal{A}'}{\mathcal{A}} = \frac{A'}{A} - \frac{2r_{*0}}{r^2}\left\{\frac{1 + \alpha M^2}{1 + 4\alpha\frac{r_{*0}}{r}}\right\} \tag{27}$$

On using the boundary condition (7), equation (27) leads to,

$$\mathcal{A}(r) = A(r)e^{-2r_{*0}\int_{r_0}^{r}\left\{\frac{1+\alpha M^2}{1+4\alpha\frac{r_{*0}}{r}}\right\}\frac{1}{r^2}\,dr}. \tag{28}$$

(28) indicates, for the polytropic wind, the effective de Laval nozzle cross-sectional area $\mathcal{A}(r)$ obtained from a *renormalization* of the actual wind channel cross-sectional area $A(r)$ *via* a multiplicative correction incorporating the solar gravity in the Parker hydrodynamic polytropic solar wind model.

Furthermore, putting $A(r) = 4\pi r^2$, we have from (28),

$$\mathcal{A}'(r) = 8\pi\left[r - r_{*0}\left(\frac{1 + \alpha M^2}{1 + 4\alpha\frac{r_{*0}}{r}}\right)\right]e^{-2r_{*0}\int_{r_0}^{r}\left(\frac{1+\alpha M^2}{1+4\alpha\frac{r_{*0}}{r}}\right)\frac{1}{r^2}\,dr} \tag{29}$$

(29) implies,

$$\mathcal{A}'(r) \lesseqgtr 0, \quad \text{if} \quad r \lesseqgtr r_* \tag{30}$$

where,

$$r_* \equiv r_{*0}(1 - 3\alpha). \tag{31}$$

In addition, noting from (29),

$$\mathcal{A}''(r) = 8\pi\left[1 - \frac{r_{*0}}{r}\left(\frac{1 + \alpha M^2}{1 + 4\alpha\frac{r_{*0}}{r}}\right)\right]e^{-2r_{*0}\int_{r_0}^{r}\left(\frac{1+\alpha M^2}{1+4\alpha\frac{r_{*0}}{r}}\right)\frac{1}{r^2}\,dr}$$

$$+8\pi\left[r-r_{*0}\left(\frac{1+\alpha M^2}{1+4\alpha\frac{r_{*0}}{r}}\right)\right]\left[-\frac{2r_{*0}}{r^2}\left(\frac{1+\alpha M^2}{1+4\alpha\frac{r_{*0}}{r}}\right)\right]e^{-2r_{*0}\int_{r_0}^{r}\left(\frac{1+\alpha M^2}{1+4\alpha\frac{r_{*0}}{r}}\right)\frac{1}{r^2}\,dr} \tag{32}$$

and using the result (Shivamoggi and Pohl [18]),

$$r=r_*:\quad M^2=1,\quad \frac{dM^2}{dr}=0$$

we obtain,

$$\mathcal{A}''(r_*)=8\pi\left(\frac{1+5\alpha}{1+\alpha}\right)e^{-2r_{*0}\int_{r_0}^{r_*}\left(\frac{1+\alpha M^2}{1+4\alpha\frac{r_{*0}}{r}}\right)\frac{1}{r^2}\,dr}>0. \tag{34}$$

Thus,

$$r\approx r_*:\mathcal{A}(r)\approx\mathcal{A}(r_*)+\frac{1}{2}\mathcal{A}''(r_*)(\Delta r)^2 \tag{35}$$

which, on using (34), confirms that the effective de Laval nozzle, for a polytropic wind, exhibits again a minimum cross-sectional area at the Parker sonic critical point, as anticipated.

In order to further see further physical implications of the above results, note that the hydrostatic force balance, from equation (2), gives:

$$-\frac{dp_h}{dr}-\frac{GM_s}{r^2}\rho_h=0. \tag{36}$$

On using (19), equation (36) leads to,

$$\frac{1}{\rho_h}\frac{d\rho_h}{dr}=-\frac{1}{r^2}\left(\frac{GM_s}{a^2}\right). \tag{37a}$$

Rewriting equation (37a) as,

$$\frac{1}{\rho_h}\frac{d\rho_h}{dr}=-\frac{2}{r^2}\left(\frac{GM_s}{2a_0^2}\right)\left(\frac{a_0^2}{a^2}\right), \tag{37b}$$

and using (21)-(23), we obtain,

$$\frac{1}{\rho_h}\frac{d\rho_h}{dr}=-\frac{2}{r^2}r_{*0}\left\{\frac{1+\alpha M^2}{1+4\alpha\frac{r_{*0}}{r}}\right\}. \tag{38}$$

Using the boundary condition (15) at the surface of the sun, equation (38) yields,

$$\rho_h=\rho_0 e^{-2r_{*0}\int_{r_0}^{r}\left\{\frac{1+\alpha M^2}{1+4\alpha\frac{r_{*0}}{r}}\right\}\frac{1}{r^2}\,dr}. \tag{39}$$

Using (39), (28) may be rewritten as,

$$\mathcal{A}(r)=A(r)\left[\frac{\rho_h(r)}{\rho_0}\right] \tag{40}$$

which is identical to the result(17), deduced before for the isothermal gas.

(40) implies that the correction factor needed to *renormalize* the actual polytropic-wind channel area to incorporate the solar gravity geometrically is precisely the *polytropic Chapman hydrostatic* density profile ρ_h/ρ_0, as in the isothermal gas case. So, Chapman's hydrostatic solar wind mechanism [4] appears to continue to be operative, on a global level (not just locally near the coronal base), in Parker's hydrodynamic solar wind model [6], even in the polytropic case, hence indicating the robustness of this result.

5 Discussion

Motivated by the strong binding of the corona by the solar gravity, Chapman [4] argued that the corona is governed by a near hydrostatic force balance condition, and hence gave a hydrostatic model for the static corona. Parker [6] pointed out that Chapman's *hydrostatic* model [4] becomes inaccurate away from the sun because the radial coronal flow becomes non-negligible, and gave a *hydrodynamic* model for the solar wind, to supersede Chapman's hydrostatic model [4], and reduce to the latter near the coronal base ($r \ll r_*$), as expected. However, the numerical calculations of Lamers and Casinelli [5] showed that the density profiles given by Chapman's hydrostatic model [4] are almost identical to those given by Parker's hydrodynamic model [6] (corresponding to the same temperature) in the *whole* subcritical region ($r \lesssim r_*$). In this paper, we have therefore investigated the *global* role of Chapman's hydrostatic solar wind mechanism [4] in Parker's hydrodynamic solar wind model [6]. We have accomplished this by using the *de Laval nozzle analogy* (Clauser [14], Parker [15]) for Parker's hydrodynamic solar wind model [6], and have shown that the action of solar gravity in Parker's hydrodynamic solar wind model [6] is geometrically equivalent to renormalization of the wind channel area, which is described precisely by Chapman's hydrostatic density profile [2]. So, Chapman's hydrostatic solar wind mechanism [4] appears to continue to be operative on a *global* level (not just locally near the coronal base) in Parker's hydrodynamic solar wind model [6], the effects of solar gravity in Parker's hydrodynamic solar wind model [6] being essentially encapsulated by Chapman's hydrostatic model [4]. The robustness of these results is confirmed by considering both isothermal gas and polytropic gas models (Parker [16], Holzer [17], Shivamoggi and Pohl [18]) for the solar wind.

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[2]The renormalization concept is standard practice in many-body physics. One example concerns *Coulomb interactions* in a plasma. A test charge is introduced in a plasma *polarizes* it and acquires a *shielding* cloud. It then becomes electrically invisible outside the cloud, and behaves like a neutral particle. So, the dielectric effects of a test charge may be transformed away *via* the electrostatic *shielding* process (Bellan [20])

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